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STABILITY OF PLANE-PARALLEL CONVECTIVE MOTION WITH RESPECT TO SPATIAL PERTURBATIONS

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The Squire transformation, known in the theory of hydrodynamic stability [1, 2], makes it possible to reduce the problem of a plane-parallel isothermal motion with respect to spatial perturbations to a problem of plane perturbations.

Formulas derived by Squire for the transformation of the Reynolds and the wave numbers allow the derivation of complete information on stability from a solution of the two-dimensional boundary value problem of Orr-Sommerfeld. It was found that plane perturbations are more dangerous because smaller (as compared to spatial perturbations) critical Reynolds numbers correspond to them.

The problem becomes more complicated in the case of a nonisothermal plane-parallel flow. The stability of a plane Poisuille flow between horizontal parallel planes heated to different temperatures was considered in [3]. A transformation similar to that of Squire is applicable in this case also, but contrary to the isothermal case, the spatial perturbations at certain specific values of parameters are here relatively more dangerous.

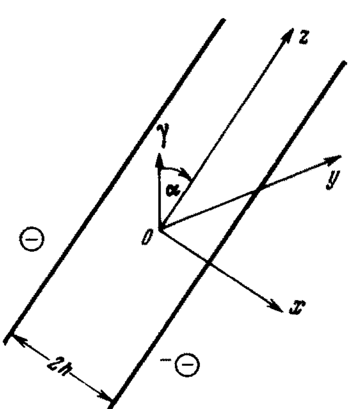

Fig. 1

The stability relative to spatial perturbations of free stationary convective motions (due to temperature nonuniformity) between infinite parallel planes, heated to different temperatures and arbitrarily orientated in the gravitational field, is considered below (Fig. 1).

Transformations of the Grashof and wave numbers, and of the angle of the layer with the vertical are derived, thus reducing the problem of stability with respect to normal spatial perturbations to the equivalent problem of plane perturbations. As the result of these transformations together with stability investigations with respect to plane perturbations [4], diagrams of convective flow stability with respect to three-dimensional perturbations were obtained.

It follows from these results that, when the layer is vertical, or in the case of an inclined layer with its upper plane at the higher temperature, plane perturbations are more dangerous. If, however, the lower plane is at the higher temperature, two instability mechanisms are at work: the hydrodynamic instability due to the effect of two opposite convection streams and the convective (the Rayleigh) instability of the fluid heated from below. The question whether the plane, or the three-dimensional perturbations are in this case more dangerous is essentially determined by two parameters: angle of inclination of the layer, and the Prandtl number.

1. Let us consider a layer of fluid bounded by parallel infinite planes $x = \pm h$ maintained at constant temperatures $\mp \Theta$ (see Fig. 1 on which the coordinate axes are shown). Under these conditions a stationary closed plane-parallel flow with a third power velocity profile and linear temperature distribution will be generated in the fluid

$$v_0 = {}^1/_6\,(x^3 - x)\cos\alpha = f(x)\cos\alpha, \qquad T_0 = -x \tag{1.1}$$

Here, and in the following, dimensionless parameters are used with the same reference dimensions as in [4].

Equations of small perturbations $(\mathbf{v}, T, p)$ of basic motion are of the form

$$\frac{\partial \mathbf{v}}{\partial t} + G\,[(\mathbf{v}\nabla)\,\mathbf{v}_0 + (\mathbf{v}_0\nabla)\,\mathbf{v}] = -\nabla p + \Delta\mathbf{v} + T\boldsymbol{\gamma} \tag{1.2}$$

$$\frac{\partial T}{\partial t} + G\,[\mathbf{v}\nabla T_0 + \mathbf{v}_0\nabla T] = \frac{1}{P}\Delta T, \qquad \operatorname{div}\mathbf{v} = 0 \tag{1.3}$$

$$\left(G = \frac{g\beta\Theta h^3}{\nu^2},\ P = \frac{\nu}{\chi}\right)$$

Here $\boldsymbol{\gamma}$ is the unit vector directed vertically upwards, and G and P are respectively the Grashof and the Prandtl numbers.

At the layer boundaries the velocity and temperature perturbations vanish

$$\mathbf{v} = 0, \ \ T = 0 \qquad (x = \pm 1) \tag{1.4}$$

We introduce normal spatial perturbations v_x, v_y, v_z, T, p dependent on time and coordinates y and z in accordance with law $\exp[-\lambda t + i(k_y y + k_z z)]$, where λ is a decrement, k_y and k_z are the wave numbers defining the perturbation periodicity along axes y and z, respectively. From (1.2)–(1.4) we obtain the following amplitude equations (the prime denotes differentiation with respect to the lateral coordinate x):

$$-\lambda v_x + ik_z G\cos\alpha\, f v_x = -p' + (v_x'' - k^2 v_x) - \sin\alpha\, T \tag{1.5}$$

$$-\lambda v_y + ik_z G\cos\alpha f v_y = -ik_y p + (v_y'' - k^2 v_y) \tag{1.6}$$

$$-\lambda v_z + ik_z G\cos\alpha f v_z + G\cos\alpha\, f' v_x =$$

$$= -ik_z\, p + (v_z'' - k^2 v_z) + \cos\alpha T \tag{1.7}$$

$$-\lambda T + ik_z G\cos\alpha\, fT + GT_0' v_x = P^{-1}(T'' - k^2 T) \tag{1.8}$$

$$v_x' + i(k_y v_y + k_z v_z) = 0 \qquad (k^2 = k_y^2 + k_z^2) \tag{1.9}$$

and the boundary conditions

$$v_x = v_y = v_z = T = 0 \quad (x = \pm 1) \tag{1.10}$$

We shall now formulate the boundary value problem of normal plane perturbation amplitudes. These perturbations are of the form

$$(v_{1x}, v_{1z}, T_1, p_1) \sim \exp(-\lambda_1 t + ik_1 z)$$

(All of the unknown functions and parameters, related to the problem of plane perturbations, will be denoted in the following by subscript 1).

The equations of plane perturbation amplitudes may be derived from (1.5)–(1.9) by setting $v_y = 0$ and $k_y = 0$

$$-\lambda_1 v_{1x} + ik_1 G_1 \cos\alpha_1 f v_{1x} = -p_1' + (v_1''_x - k^2_1 v_{1x}) - \sin\alpha_1 T_1 \quad (1.11)$$

$$-\lambda_1 v_{1z} + ik_1 G_1 \cos\alpha_1 f v_{1z} + G_1 \cos\alpha_1 f' v_{1x} =$$

$$= -ik_1 p_1 + (v_{1z}'' - k_1^2 v_{1z}) + \cos\alpha_1 T_1 \quad (1.12)$$

$$-\lambda_1 T_1 + ik_1 G_1 \cos\alpha_1 f T_1 + G_1 T_0' v_{1x} = P_1^{-1}(T_1'' - k_1^2 T_1) \quad (1.13)$$

$$v_{1x}' + ik_1 v_{1z} = 0 \quad (1.14)$$

The boundary conditions for plane perturbations are

$$v_{1x} = v_{1z} = T_1 = 0 \quad (x = \pm 1) \quad (1.15)$$

We shall now show the existence of transformations by means of which the spatial problem (1.5)–(1.10) may be reduced to a plane problem (1.11)–(1.15).

If we assume

$$v_x = v_{1x}, \quad k_y v_y + k_z v_z = k_1 v_{1z} \quad (1.16)$$

then the continuity equation (1.9) takes the form of (1.14).

In order to transform Eq. (1.5) into (1.11) we have to supplement (1.16) by the following transformations

$$\lambda = \lambda_1, \; k_z G \cos\alpha = k_1 G_1 \cos\alpha_1, \quad p = p_1$$

$$\sin\alpha T = \sin\alpha_1 T_1, \qquad k^2 = k_y^2 + k_z^2 = k_1^2 \quad (1.17)$$

Combining (1.6) and (1.7) yields (1.12), if we specify additionally the condition

$$k_z \cos\alpha\, T = k_1 \cos\alpha_1 T_1 \quad (1.18)$$

Finally, the identification of the heat conduction Eqs. (1.8) and (1.13) imposes two more conditions

$$\sin\alpha\, G = \sin\alpha_1 G_1, \qquad P = P_1 \quad (1.19)$$

Functions v_{1x}, v_{1z}, T_1 defined by relationships (1.16)–(1.19) satisfy boundary conditions (1.15).

The system of Eqs. (1.16)–(1.19) is consistent, and yields two transformations which reduce the spatial perturbation problem to the corresponding problem of plane perturbations (*).

In particular, the Grashof and the wave numbers and the layer inclination angle are transformed as follows:

$$G = G_1 \sqrt{\sin^2\alpha_1 + (k_1/k_z)^2 \cos^2\alpha_1}$$

$$k_1 = \sqrt{k_y^2 + k_z^2} \qquad k_1 \operatorname{tg}\alpha = k_z \operatorname{tg}\alpha_1 \quad (1.20)$$

Thus, the critical number G of three-dimensional perturbations at wave numbers k_y

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*) The spectrum branch corresponding to the v_y-perturbations is lost in this reduction to a plane problem. It is readily ascertained by Eq. (1.6) that such perturbations, as in the case considered by Squire [1], become attenuated independently of the basic velocity profile.

and k_z in the case of a layer inclined at angle α to the vertical may be determined from formula (1.20), if the critical Grashof number G_1 for plane perturbations at wave number k_1 in a layer inclined at angle α_1 (different from α) is known.

2. We shall consider two particular cases.

If the layer is horizontal ($\alpha = \pm 90°$), the stationary motion velocity vanishes (see (1.1)), and the problem reduces to the determination of conditions of equilibrium stability of the horizontal fluid layer heated either from below ($\alpha = -90°$) or from above ($\alpha = 90°$). It follows from (1.20) that in this case $\alpha_1 = \pm 90°$ and $G = G_1$.

Thus, the critical numbers for spatial and plane perturbations are the same (the known degeneration of the Rayleigh instability limit of convection "cells" and "ridges").

In the case of a vertical layer ($\alpha = 0$) we obtain from formulas (1.20)

$$\alpha_1 = 0, \quad G = G_1 \; k_1 \, / \, k_z \tag{2.1}$$

As $k_z / k_1 = k_z / (k_y^2 + k_z^2)^{1/2} \leqslant 1$, hence $G \geqslant G_1$, i.e. in this case, as in that of isothermal motion, higher critical Grashof numbers correspond to three-dimensional perturbations, and consequently, plane perturbations ($k_y = 0$) are the more dangerous.

We shall now consider the case of an arbitrary orientation of the layer. The solution of the problem of convective motion stability with respect to plane perturbations was derived in [4], where the dependence of the minimum critical Grashof numbers G_{1m} on the angle of inclination α_1 and the Prandtl number are given. Critical parameters of spatial perturbations can be evaluated and defined by using formulas (1.20). For evaluation it is convenient to assume parameter $a \equiv k_z / k_1$ which varies within limits $0 \leqslant a \leqslant 1$, as fixed. The value $a = 1$ corresponds to plane perturbations ($k_y = 0$ relates to ridges with horizontal axes). The value $a = 0$ corresponds to perturbations periodically dependent on the horizontal coordinate y and independent of z ($k_z = 0$ relates to ridges the axes of which are parallel to the basic motion velocity).

In the case of spatial perturbations the relationship $G = G(k_y, k_z)$ defines the neutral surface (an analog of the neutral curve $G_1(k_1)$ of plane perturbations). The intersection of this surface with the plane passing through axis G corresponds to the fixed value of a, and the intersection curve corresponds to the minimum G_m which depends on α and a. For given values of α and a the critical numbers G and G_1 differ in accordance with (1.20) (for all wave numbers) by one and the same factor $(\sin^2\alpha_1 + a^{-2} \cos^2\alpha_1)^{1/2}$. Thus, operator (1.20) transforms the minimum point G_{1m} of the plane problem neutral curve into the same intersection point G_m of the neutral surface.

Computation results of minimum critical numbers G_m of spatial perturbations are shown on Figs. 2 to 4 for various values of parameter a. These computations were made for three values of the Prandtl number, using formulas (1.20) and the relationship $G_1(k_1)$ of [4].

The graphs shown in Figs. 2 and 3 relate to Prandtl numbers $P = 5$ and $P = 1$. It will be seen from these that for $\alpha \geqslant 0$ (a vertical layer, or an inclined one with its upper surface at the higher temperature) the $G_m(\alpha)$ plane perturbations ($a = 1$) are the more dangerous. If $\alpha < 0$ (an inclined layer with its lower surface at the higher temperature), the spatial perturbations become more dangerous in a wide field of variation of α, with G_m reaching its absolute minimum at $a = 0$. As already indicated, perturbations periodic along the horizontal y-coordinate and independent of z ($k_z = 0$), correspond to the latter value of a.

The critical number $G_m(\alpha)$ may be found for this critical limit case directly from the general boundary value problem of spatial perturbations, as defined by (1.5)–(1.10). Assuming $k_z = 0$ we obtain from Eqs. (1.5), (1.6), (1.8) and (1.9)

$$-\lambda v_x = -p' + (v_x'' - k_y^2 v_x) - \sin\alpha T, \qquad -\lambda v_y = -ik_y p + (v_y'' - k_y^2 v_y)$$
$$v_x' + ik_y v_y = 0, \qquad -\lambda T + GT_0' v_x = P^{-1}(T'' - k_y^2 T) \tag{2.2}$$

with homogeneous boundary conditions

$$v_x = v_y = T = 0 \quad (x = \pm 1) \tag{2.3}$$

This boundary value problem is independent of the basic motion profile and coincides with the Rayleigh problem of perturbations in a stationary fluid layer (*). The critical Grashof number, derived in the usual manner from the condition $\lambda = 0$ by minimizing $G(k_y)$, is expressed by

$$-G_m P \sin\alpha = 1708/16 \tag{2.4}$$

(Factor $^1/_{16}$ is consequent to the selection of the layer half-width and half the temperature difference at its boundaries, as unit of distance and temperature respectively). Thus, for perturbations at $k_z = 0$ we have

$$G_m = \frac{106.7}{-P\sin\alpha} \tag{2.5}$$

Spatial perturbations at $k_z = 0$ correspond to the absolute minimum G_m in the angle range $-90° < \alpha < \alpha_*$, where α_* is defined by the intersection of curves $G_m(\alpha)$ corresponding to parameters $a = 0$ and $a = 1$. The value of α^* depends on the Prandtl number, thus, for $P = 5$ and $P = 1$ we have respectively $\alpha_* = -2.5°$ and $\alpha_* = -13°$. With decreasing Prandtl number the intersection point moves to the left.

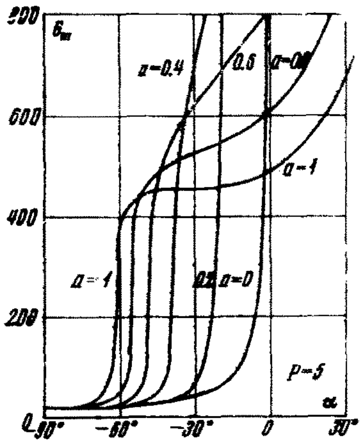

Fig. 2

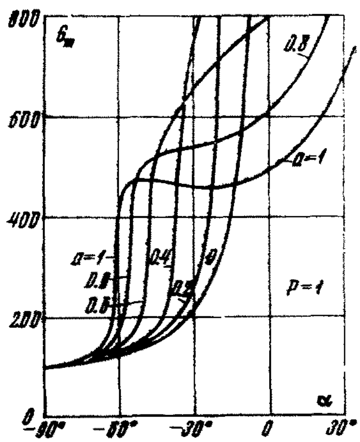

Fig. 3

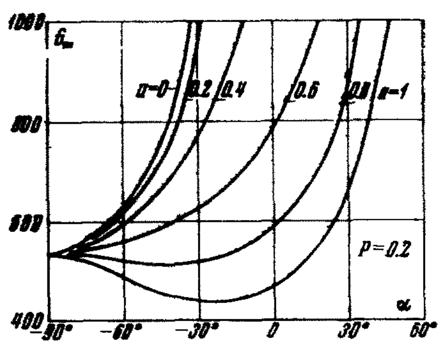

Fig. 4

*) If $k_z = 0$, the perturbations are not plane. The velocity component v_z is not zero and may be found from Eq. (1.7).

tending to $\alpha_* = -90°$. Thus, at sufficiently small Prandtl numbers (see Fig. 4, where $P = 0.2$) the absolute minimum moves towards perturbations at $a = 1$ (plane perturbations) for all angles $\alpha > -90°$ (at $\alpha = -90°$ the previously mentioned degeneration occurs).

It has been already shown [4], that crisis of a convective flow at various angles of inclination is due to two physically different mechanisms of instability. In the region of angles close to $-90°$ (the Rayleigh region) the crisis is the result of the convective instability of the fluid heated from below, while outside of the Rayleigh region it is caused by the hydromechanical mechanism of the instability of the two contrary convective streams.

The results of the present investigation of stability with respect to spatial perturbations confirm the change of the instability mechanism by a change of the angle of inclination. In the Rayleigh region the instability is, in fact, due to spatial perturbations when $a = 0$; the crisis of such perturbations is related to the unstable temperature stratification only, and is independent of the fluid stationary motion (the velocity of this motion at angles close to $-90°$ is low, and this motion itself is hydrodynamically stable). In the case of a vertical layer, and even more so at $\alpha > 0$, this instability can only be caused by the instability of the contrary convective streams, and then the plane perturbations ($a = 1$) are the more dangerous, as in the case of plane-parallel flows of an isothermal fluid.

If the Prandtl number is small (low viscosity), even a small deviation of the layer from the horizontal position ($\alpha \gtrsim -90°$) results in an intensive convective motion, and in this case, even at angles close to $-90°$ (i.e. in the Rayleigh region), the instability is of a hydrodynamic nature, and related to the development of plane perturbations.

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